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On Higher-spin Generalisations of String Theory

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ABSTRACT

We construct BRST operators for certain higher-spin extensions of the Virasoro algebra, in which there is a spin- s gauge field on the world sheet, as well as the spin-2 gauge field corresponding to the two-dimensional metric. We use these BRST operators to study the physical states of the associated string theories, and show how they are related to certain minimal models.

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1. Introduction

Two-dimensional gauge theories are the progenitors of string theories. By gauging the Virasoro algebra, realised as a semi-local symmetry of a set of free scalar fields, one is led to the usual bosonic string. If one gauges the supersymmetric extension of the Virasoro algebra, one obtains a supersymmetric string theory. Since there exist many other extensions of the Virasoro algebra, in which one adds currents with spins greater than two, it is natural to enquire whether these too can give rise to interesting generalisations of string theory. Such algebras are generically called W algebras, and the resulting string theories are called W strings. A general feature of the W algebras is that they are non-linear, and this leads to certain complications both of a conceptual and a technical nature when one builds the corresponding string theories.

The simplest example of a W algebra is the W_3 algebra of Zamolodchikov [1]. This contains a spin-3 primary current W in addition to the energy-momentum tensor T . The idea of gauging the algebra, to obtain a two-dimensional theory of W_3 gravity, was first put forward in [2]. This leads naturally to the idea of building a W_3 string [3,4,5]. The first requirement for building such a string is an anomaly-free theory of W_3 gravity. In [5], it was shown how such a theory may be built, using standard BRST techniques. The BRST operator Q_B for the W_3 algebra had been found in [6,7], where it was shown that nilpotency demands that the matter currents T and W should generate the W_3 algebra with central charge $c = 100$. Multi-scalar realisations were found in [8], in terms of a set of fields (φ, X^μ) . The scalar φ plays a special rôle, which we shall explain later, whilst the d scalars X^μ appear in T and W only *via* their energy-momentum tensor T^{eff} . The fields X^μ will acquire the interpretation of coordinates on an effective target spacetime. For T and W to generate the W_3 algebra with $c = 100$, it is necessary for T^{eff} to generate the Virasoro algebra with central charge $c^{\text{eff}} = \frac{51}{2}$ [4,5]. Thus a background charge is needed regardless of the dimension d of the spacetime described by X^μ , and so there is no notion of a “critical dimension” for the theory.

The spectrum of physical states can be studied most elegantly in the BRST formalism: A physical state $|\chi\rangle$ is one that is annihilated by the BRST operator but is not BRST trivial. Thus, $Q_B|\chi\rangle = 0$ but $|\chi\rangle \neq Q_B|\psi\rangle$ for any state $|\psi\rangle$. It was shown recently that a dramatic simplification of the BRST operator, and the physical states, can be achieved by performing a redefinition under which the ghost fields (b, c) for the spin-2 current and (β, γ) for the spin-3 current are mixed with the special scalar φ mentioned above [9]. In terms of the redefined fields, the BRST operator becomes [9]

$$Q_B = Q_0 + Q_1, \tag{1.1}$$

$$Q_0 = \oint dz c \left(T^{\text{eff}} + T_\varphi + T_{\gamma, \beta} + \frac{1}{2} T_{c, b} \right), \tag{1.2}$$

$$Q_1 = \oint dz \gamma \left((\partial\varphi)^3 + 3\alpha \partial^2\varphi \partial\varphi + \frac{19}{8} \partial^3\varphi + \frac{9}{2} \partial\varphi \beta \partial\gamma + \frac{3}{2} \alpha \partial\beta \partial\gamma \right), \tag{1.3}$$

where the energy-momentum tensors are given by

$$T_\varphi \equiv -\frac{1}{2}(\partial\varphi)^2 - \alpha \partial^2\varphi, \quad (1.4)$$

$$T_{\gamma,\beta} \equiv -3\beta\partial\gamma - 2\partial\beta\gamma, \quad (1.5)$$

$$T_{c,b} \equiv -2b\partial c - \partial b c, \quad (1.6)$$

$$T^{\text{eff}} \equiv -\frac{1}{2}\partial X^\mu \partial X^\nu \eta_{\mu\nu} - i a_\mu \partial^2 X^\mu. \quad (1.7)$$

The background charge α for the scalar φ is given by $\alpha^2 = \frac{49}{8}$, and the background-charge vector a_μ is chosen so that $d - 12a_\mu a^\mu = \frac{51}{2}$. The BRST operator is graded, with $Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0$.

It was found in [10,11,9] that the physical states of the W_3 string comprise four basic sectors. The first three sectors are described by physical operators V of the form [9]

$$V_\Delta = c U(\beta, \gamma, \varphi) V^{\text{eff}}(X), \quad (1.8)$$

where $[Q_0, V_\Delta] = 0$, and the Q_1 constraint reduces to $[Q_1, U(\beta, \gamma, \varphi)] = 0$. These conditions imply that V^{eff} is an effective physical operator in the spacetime X^μ , corresponding to effective physical states $|\text{phys}\rangle_{\text{eff}} \equiv V^{\text{eff}}(X(0))|0\rangle$ satisfying the highest-weight conditions

$$\begin{aligned} L_n^{\text{eff}} |\text{phys}\rangle_{\text{eff}} &= 0, & n > 0, \\ (L_0^{\text{eff}} - \Delta) |\text{phys}\rangle_{\text{eff}} &= 0. \end{aligned} \quad (1.9)$$

The intercept Δ takes the values $\Delta = 1, \frac{15}{16}, \frac{1}{2}$ in the three sectors. The fourth sector of physical states is described by operators of the form [9]

$$V_0 = c U_1(\beta, \gamma, \varphi) + U_2(\beta, \gamma, \varphi). \quad (1.10)$$

These correspond to discrete states, with zero momentum in the effective spacetime. The operators V_Δ in the first three sectors can be viewed as representatives of the physical operators for effective Virasoro strings with the three intercept values Δ given above, whilst all the discrete operators V_0 can be viewed as representatives of the identity operator in the effective spacetime. Many examples for all four sectors, up to level 9 in excitations of the (φ, β, γ) system, are given in [9].

A procedure for computing scattering amplitudes for the physical states of the W_3 string was presented in [11,9]. It was found by studying many examples that the amplitudes are characterised by the weights Δ of the physical operators V_Δ appearing in the correlation functions. If one associates the operators V_1 , $V_{15/16}$ and $V_{1/2}$ respectively with the weight $\{0, \frac{15}{16}, \frac{1}{2}\}$ primary fields $\{1, \sigma, \varepsilon\}$ of the Ising model, then the pattern of vanishing and non-vanishing three-point functions is in one-to-one correspondence with the fusion rules of the Ising model [11]. The four-point and higher-point functions exhibit duality and factorisation

properties that are consistent with these underlying three-point functions. In fact the picture that emerges is that the W_3 string, and its interactions, can really be viewed as being described by a special case of an ordinary conformal field theory in which one takes the tensor product of the ($c = \frac{1}{2}$) Ising model with a $c = \frac{51}{2}$ energy-momentum tensor for d scalar fields X^μ with a background charge. (See [9,12] for a discussion of a minor technicality associated with the need to include the discrete operators V_0 as representatives of the identity in correlation functions in order to reproduce the full set of correlation functions for the Ising model.) From the effective spacetime point of view, the scattering amplitudes for the W_3 string, obtained using the procedure introduced in [11,9], coincide with those for the tensor product of the Ising model and a $c = \frac{51}{2}$ Virasoro string, which was studied in [13] by using the sophisticated “Group Theoretic Method.”

2. Higher-spin generalisations

A natural generalisation of the W_3 string is to study the W_N string, obtained by gauging the W_N algebra. This has currents of spins 3, 4, \dots , N in addition to the energy-momentum tensor T . Since the BRST operator for the W_N algebra is not known, except for the case of W_3 , a complete investigation for general N is not possible. However, by making certain plausible assumptions [4] the spectrum of physical states with standard ghost structure has been determined [14]. The indications are that the full spectrum of physical states will coincide with those that would be obtained by taking the tensor product of the N 'th unitary Virasoro minimal model with an energy-momentum tensor T^{eff} , with central charge given by $c = 26 - \left[1 - \frac{6}{N(N+1)}\right]$, for the target spacetime fields X^μ .

A simpler possibility for higher-spin generalisations of the W_3 string is to consider the case of an algebra involving just two currents, namely the energy-momentum tensor T and a spin- s current W . The case $s = 3$ corresponds to the W_3 algebra itself. It should be possible to realise such algebras in terms of a scalar field φ and an energy-momentum tensor T^{eff} for a set of scalars X^μ , analogous to the realisation of W_3 . One would expect that the BRST operator for such an algebra could again be simplified considerably by the kind of redefinitions amongst the ghost and φ field that were obtained for W_3 in [9], and which lead to the form (1.1–1.3) for its BRST operator. In fact we can take a short cut to the construction of the BRST operator in the redefined formalism by simply making an ansatz that appropriately generalises (1.1–1.3) for the BRST operator, and then requiring that it be nilpotent. Using the same notation as for the W_3 case, we introduce the usual (b, c) ghost system for the spin-2 current, and the (β, γ) ghost system for the spin- s current. Note that β therefore has spin s , and γ has spin $(1 - s)$. Thus we begin by writing

$$Q_B = Q_0 + Q_1, \tag{2.1}$$

$$Q_0 = \oint dz \, c \left(T^{\text{eff}} + T_\varphi + T_{\gamma, \beta} + \frac{1}{2} T_{c, b} \right), \tag{2.2}$$

$$Q_1 = \oint dz \gamma F(\beta, \gamma, \varphi), \quad (2.3)$$

where the energy-momentum tensors are given by

$$T_\varphi \equiv -\frac{1}{2}(\partial\varphi)^2 - \alpha \partial^2 \varphi, \quad (2.4)$$

$$T_{\gamma,\beta} \equiv -s \beta \partial\gamma - (s-1) \partial\beta \gamma, \quad (2.5)$$

$$T_{c,b} \equiv -2 b \partial c - \partial b c, \quad (2.6)$$

$$T^{\text{eff}} \equiv -\frac{1}{2} \partial X^\mu \partial X^\nu \eta_{\mu\nu} - i a_\mu \partial^2 X^\mu. \quad (2.7)$$

The operator $F(\beta, \gamma, \varphi)$ has spin s and ghost number zero, generalising the quantity appearing in (1.3) in the case of W_3 .

We expect on general grounds that operators Q_0 and Q_1 defined as above should exist, satisfying $Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0$. One way to see this is to note that one expects that a BRST operator for W_N , with a grading $\tilde{Q}_B = \sum_{i=0}^{N-2} \tilde{Q}_i$, $\{\tilde{Q}_i, \tilde{Q}_j\} = 0$, should exist, generalising the grading of the W_3 case discussed in section 1 [15]. A realisation of W_N can be given in terms of the currents of W_{N-1} and one additional scalar, φ [8,4,14]. The highest term in \tilde{Q}_B , namely \tilde{Q}_{N-2} , would involve just the field φ and the ghost system for the spin- N current of W_N [16]. Thus the nilpotency of \tilde{Q}_{N-2} would be a property that is independent of the nature of the currents of lower spin, and so the form of Q_1 in our BRST operator for the spin-2 plus spin- s system will be the same as that of \tilde{Q}_{N-2} for the BRST operator of W_s . It was shown in [14] that the scalar φ involved in the realisation of W_s currents in terms of W_{s-1} currents must have a background charge α given by

$$\alpha^2 = \frac{(s-1)(2s+1)^2}{4(s+1)} \quad (2.8)$$

in order to achieve the correct central charge for criticality. Thus we expect that for each s there should be at least a nilpotent Q_B given by (2.1–2.3) with the parameter α given by (2.8). In fact, as we shall see, we find by demanding the nilpotency of Q_B defined in (2.1) that in addition to the above solution, with α given by (2.8), there can be further solutions, with different operators $F(\beta, \gamma, \varphi)$ and different values of α . These seem to be unrelated to the W_s algebra.

Owing to the grading of Q_B , the conditions for its nilpotency separate into the conditions $Q_0^2 = 0$, $\{Q_0, Q_1\} = 0$ and $Q_1^2 = 0$. The first of these is easily analysed in general. Nilpotency of Q_0 is achieved provided that the total central charge vanishes, *i.e.*

$$0 = -26 - 2(6s^2 - 6s + 1) + 1 + 12\alpha^2 + c^{\text{eff}}. \quad (2.9)$$

The effective central charge c^{eff} is given by $c^{\text{eff}} = d - 12a_\mu a^\mu$. For the remaining nilpotency conditions, we shall carry out a case-by-case analysis for $s = 4, 5$ and 6 .

We begin with the case $s = 4$. The most general ansatz for the spin-4 ghost number zero operator $F(\beta, \gamma, \varphi)$ in (2.3) has twelve terms, two of which can be eliminated since they correspond to total derivatives. The condition $\{Q_0, Q_1\} = 0$ then leads to a system of linear equations for the coefficients of the remaining terms, and the condition $Q_1^2 = 0$ gives rise to quadratic equations for the coefficients. We find two distinct solutions; one with $\alpha^2 = \frac{243}{20}$ which corresponds to the expected $s = 4$ case of (2.8) associated with the W_4 algebra, and the other with $\alpha^2 = \frac{361}{30}$. For the first solution, with $\alpha^2 = \frac{243}{20}$, we find

$$F(\beta, \gamma, \varphi) = (\partial\varphi)^4 + 4\alpha \partial^2\varphi (\partial\varphi)^2 + \frac{41}{5}(\partial^2\varphi)^2 + \frac{124}{15}\partial^3\varphi \partial\varphi + \frac{46}{135}\alpha \partial^4\varphi \\ + 8(\partial\varphi)^2 \beta \partial\gamma - \frac{16}{9}\alpha \partial^2\varphi \beta \partial\gamma - \frac{32}{9}\alpha \partial\varphi \beta \partial^2\gamma - \frac{4}{5}\beta \partial^3\gamma + \frac{16}{3}\partial^2\beta \partial\gamma. \quad (2.10)$$

For the second solution, with $\alpha^2 = \frac{361}{30}$, we find the corresponding operator $F(\beta, \gamma, \varphi)$ in (2.3) is given by

$$F(\beta, \gamma, \varphi) = (\partial\varphi)^4 + 4\alpha \partial^2\varphi (\partial\varphi)^2 + \frac{253}{30}(\partial^2\varphi)^2 + \frac{39}{5}\partial^3\varphi \partial\varphi + \frac{41}{570}\alpha \partial^4\varphi \\ + 8(\partial\varphi)^2 \beta \partial\gamma - \frac{26}{19}\alpha \partial^2\varphi \beta \partial\gamma - \frac{66}{19}\alpha \partial\varphi \beta \partial^2\gamma - \frac{26}{15}\beta \partial^3\gamma + \frac{29}{5}\partial^2\beta \partial\gamma. \quad (2.11)$$

(We remind the reader that the spins of the (β, γ) ghosts depend upon the value of s ; namely β has spin s and γ has spin $(1 - s)$).

For the case of $s = 5$, we find just one solution, one with $\alpha^2 = \frac{121}{6}$ given by (2.8) with $s = 5$, namely $\alpha^2 = \frac{121}{6}$. Then $F(\beta, \gamma, \varphi)$ in (2.3) is given by

$$F(\beta, \gamma, \varphi) = (\partial\varphi)^5 + 5\alpha \partial^2\varphi (\partial\varphi)^3 + \frac{305}{8}(\partial^2\varphi)^2 \partial\varphi + \frac{115}{6}\partial^3\varphi (\partial\varphi)^2 + \frac{10}{3}\alpha \partial^3\varphi \partial^2\varphi \\ + \frac{55}{48}\alpha \partial^4\varphi \partial\varphi + \frac{251}{576}\partial^5\varphi + \frac{25}{2}(\partial\varphi)^3 \beta \partial\gamma + \frac{25}{4}\alpha \partial^2\varphi \partial\varphi \beta \partial\gamma + \frac{25}{4}\alpha (\partial\varphi)^2 \partial\beta \partial\gamma \\ + \frac{125}{16}\partial^3\varphi \beta \partial\gamma + \frac{325}{12}\partial^2\varphi \partial\beta \partial\gamma + \frac{375}{16}\partial\varphi \partial^2\beta \partial\gamma - \frac{175}{48}\partial\varphi \beta \partial^3\gamma \\ + \frac{5}{3}\alpha \partial^3\beta \partial\gamma - \frac{35}{48}\alpha \partial\beta \partial^3\gamma. \quad (2.12)$$

Finally, we present results for the case of $s = 6$. Here, we find four solutions, with $\alpha^2 = \frac{845}{28}, \frac{1681}{56}, \frac{361}{12}, \frac{5041}{168}$. The first of these is the case corresponding to the W_6 algebra, given by (2.8). For this case, we find

$$F(\beta, \gamma, \varphi) = (\partial\varphi)^6 + 6\alpha \partial^2\varphi (\partial\varphi)^4 + \frac{765}{7}(\partial^2\varphi)^2 (\partial\varphi)^2 + \frac{256}{7}\partial^3\varphi (\partial\varphi)^3 + \frac{174}{35}\alpha (\partial^2\varphi)^3 \\ + \frac{528}{35}\alpha \partial^3\varphi \partial^2\varphi \partial\varphi + \frac{18}{7}\alpha \partial^4\varphi (\partial\varphi)^2 + \frac{1514}{245}(\partial^3\varphi)^2 + \frac{2061}{245}\partial^4\varphi \partial^2\varphi + \frac{2736}{1225}\partial^5\varphi \partial\varphi \\ + \frac{142}{6125}\alpha \partial^6\varphi + 18(\partial\varphi)^4 \beta \partial\gamma + \frac{72}{5}\alpha \partial^2\varphi (\partial\varphi)^2 \beta \partial\gamma + \frac{48}{5}\alpha (\partial\varphi)^3 \partial\beta \partial\gamma \\ + \frac{216}{5}\partial^3\varphi \partial\varphi \beta \partial\gamma + \frac{1494}{35}(\partial^2\varphi)^2 \beta \partial\gamma + \frac{5256}{35}\partial^2\varphi \partial\varphi \partial\beta \partial\gamma + \frac{324}{5}(\partial\varphi)^2 \partial^2\beta \partial\gamma \\ - \frac{72}{7}(\partial\varphi)^2 \beta \partial^3\gamma + \frac{204}{175}\alpha \partial^4\varphi \beta \partial\gamma + \frac{192}{25}\alpha \partial^3\varphi \partial\beta \partial\gamma + \frac{2376}{175}\alpha \partial^2\varphi \partial^2\beta \partial\gamma \\ - \frac{144}{175}\alpha \partial^2\varphi \beta \partial^3\gamma + \frac{1296}{175}\alpha \partial\varphi \partial^3\beta \partial\gamma - \frac{576}{175}\alpha \partial\varphi \partial\beta \partial^3\gamma + \frac{1614}{175}\partial^4\beta \partial\gamma \\ - \frac{216}{35}\partial^2\beta \partial^3\gamma + \frac{144}{1225}\beta \partial^5\gamma + \frac{144}{35}\partial\beta \beta \partial^2\gamma \partial\gamma. \quad (2.13)$$

We have also solved for $F(\beta, \gamma, \varphi)$ for the other values for the background charge α , but we shall not present them explicitly here.

3. Physical states

We have seen in section 2, both from general arguments, and from explicit solutions, that there exist nilpotent BRST operators for the spin-2 plus spin- s systems of the form (2.1–2.3). We may now consider solving for physical states in the corresponding string theories, by requiring that they be annihilated by Q_B but that they should not be BRST trivial.

We begin by considering the spin-2 plus spin- s theory with the background charge α given by (2.8). In this case, the spin-2 and spin- s currents can be thought of as a subset of the spin 2, 3 \dots , s currents of the W_s algebra. It was shown in [14] that if one constructs the W_s currents in terms of φ and the currents of W_{s-1} , then the algebra of the W_{s-1} currents has the central charge conjugate to that of the lowest non-trivial minimal model of the W_{s-1} algebra (the generalisation of the Ising model), namely $c = \frac{2(s-2)}{(s+1)}$. Thus for our spin-2 plus spin- s theory, we should find that T^{eff} has central charge conjugate to that of the lowest W_{s-1} minimal model, namely

$$c^{\text{eff}} = 26 - \frac{2(s-2)}{(s+1)}. \quad (3.1)$$

Indeed this is the case, as can be seen by substituting (2.8) into (2.9).

The above discussion leads one to expect that the physical states of the spin-2 plus spin- s string theory should be described by a set of Virasoro-like strings for an energy momentum tensor T^{eff} with central charge given by (3.1), and intercepts given by $\Delta = 1 - h$, where h takes values that include those of the lowest W_{s-1} minimal model. (Since the constraints for the spin-2 plus spin- s string are fewer than in a W_s string, the physical-state conditions should be less stringent, and therefore they should admit more solutions.) We shall now discuss the examples of $s = 4, 5$ and 6 in detail, and see that indeed these expectations are fulfilled.

Consider first the spin-2 plus spin-4 string. From (3.1), we see that T^{eff} in this case has central charge $26 - \frac{4}{5}$, conjugate to the lowest W_3 minimal model, which has $c = \frac{4}{5}$. Fortuitously in this case, this central charge also coincides with a Virasoro minimal model, namely the $N = 5$ model (the three-state Potts model). Thus we would expect to find effective Virasoro strings with intercepts $\Delta = 1 - h$ for a set of h values that includes the conformal weights of the lowest W_3 minimal model, and is included in the set of weights of the $N = 5$ Virasoro minimal model. Physical states of “standard” ghost structure are described in the spin-2 plus spin-4 string by BRST-invariant operators with ghost number $G = 4$ of the form

$$U = c \partial^2 \gamma \partial \gamma \gamma e^{\mu \varphi} e^{ip \cdot X}. \quad (3.2)$$

(For convenience, we always discuss physical states that are tachyonic from the point of view of the effective spacetime. One can always replace the tachyon vertex operator $e^{ip \cdot X}$ by any excited effective physical operator, *i.e.* by an operator involving excitations of X^μ which is highest weight under T^{eff} , with the same intercept Δ as the tachyon.) Acting with Q_B of (2.1), with $F(\beta, \gamma, \varphi)$ given by (2.10), and requiring that (3.2) be annihilated, we find that the φ momentum μ in (3.2) must take one of the values $\mu = -\frac{8}{9}\alpha, -\frac{10}{9}\alpha, -\frac{26}{27}\alpha, -\frac{28}{27}\alpha$, and that correspondingly the intercept for $e^{ip \cdot X}$ takes the values $\Delta = 1, 1, \frac{14}{15}, \frac{14}{15}$. At level $\ell = 1$ in excitations of β, γ and φ , we find physical operators with ghost number $G = 3$ of the form

$$U = c \partial \gamma \gamma e^{\mu \varphi} e^{ip \cdot X}, \quad (3.3)$$

with $\mu = -\frac{2}{3}\alpha, -\frac{16}{27}\alpha, -\frac{20}{27}\alpha$, and $\Delta = \frac{3}{5}, \frac{14}{15}, \frac{1}{3}$ respectively. We have solved for all the physical operators at levels ℓ up to and including 9. The results for the φ momenta μ , and the effective intercepts Δ , are given in Table 1 below.

	G	Δ	μ (In units of $\alpha/27$)
$\ell = 0$	4	$\frac{14}{15} \quad 1$	$\{-28, -26\} \quad \{-30, -24\}$
$\ell = 1$	3	$\frac{1}{3} \quad \frac{3}{5} \quad \frac{14}{15}$	$-20 \quad -18 \quad -16$
$\ell = 2$	3	$-\frac{2}{5} \quad \frac{1}{3}$	$-18 \quad -14$
$\ell = 3$	2	$\frac{1}{3} \quad \frac{14}{15}$	$-10 \quad -8$
$\ell = 4$	2	$\frac{3}{5}$	-6
$\ell = 5$	2	$-\frac{2}{5} \quad \frac{1}{3}$	$-6 \quad -4$
$\ell = 6$	1	1	0
$\ell = 7$	1	$\frac{14}{15}$	2
$\ell = 8$	1	$\frac{14}{15}$	4
$\ell = 9$	1	$-2 \quad 1$	$0 \quad 6$

Table 1. Continuous-momentum physical operators for the spin-2 plus spin-4 string

One can see from the results that the momentum μ in the φ direction is always frozen to values that are integer multiples of $\frac{1}{27}\alpha$. If we assume that this is true in general, $\mu = \frac{k}{27}\alpha$, then it is easy to see that the mass-shell condition $-\frac{1}{2}\mu^2 - \mu\alpha + \frac{1}{2}p^2 + p \cdot a = 7 - \ell$, *i.e.* $-\frac{1}{2}\mu^2 - \mu\alpha + \Delta = 7 - \ell$, implies that a necessary condition for the existence of a solution with effective intercept Δ at level ℓ is that there exist an integer k such that

$$(k + 27)^2 = 120(\Delta + \ell) - 111. \quad (3.4)$$

We believe, for reasons that will emerge below, that the physical states that we have found explicitly, and that are tabulated in Table 1, include representatives with all the possible intercepts Δ for the spin-2 plus spin-4 string.

There are also discrete physical states in the theory, with zero momentum in the effective spacetime. One can apply (3.4), with $\Delta = 0$, to determine the levels at which such states

might arise. The result is $\ell = 1, 7, 10, 28, 34, \dots$. At $\ell = 7$ there is a solution with $k = 0$; this corresponds simply to the identity operator 1, which gives rise to the $SL(2, C)$ vacuum as physical state. At the next allowed level, $\ell = 10$, we can expect to find the analogue of the ground-ring operator that arises at level 2 in the two-scalar string [17], and at level 6 in the W_3 string [10]. One can associate a screening current S with any physical operator U , according to the prescription

$$S(w) = \oint dz b(z)U(w). \quad (3.5)$$

It was shown in [9] that by applying this to the level 6 discrete state of the W_3 string, one obtains, after dropping a total derivative term, the very simple screening current $S = \beta e^{\frac{2}{7}\alpha\varphi}$. Thus we may expect that an analogous screening current should exist here for the spin-2 plus spin-4 string. From (3.4), it should have momentum $\mu = \frac{2}{9}\alpha$, and indeed we find that such a screening current,

$$S = \beta e^{\frac{2}{9}\alpha\varphi} \quad (3.6)$$

exists (*i.e.* Q_B acting on S gives a total derivative). Note that S has ghost number $G = -1$. It satisfies

$$\{Q_B, S\} = \partial D, \quad (3.7)$$

where D is the level $\ell = 10$ discrete state mentioned above.

It was argued in [12] that one can expect the screening current analogous to (3.6) in the W_3 string to act as a generator of all the higher-level physical states, by acting on the lowest-level representatives for each effective intercept Δ with powers of the screening charge. Similarly, we can expect here that the charge constructed by integrating (3.6) can be applied to the lowest-level representative in each sector of physical states listed in Table 1 in order to generate the entire higher-level spectrum*. In view of its φ momentum $\frac{2}{9}\alpha$, and its ghost number $G = -1$, one expects that the trend for the higher-level states is for the φ momentum to increase, and the ghost number to decrease. This is analogous to what happens in the W_3 string.

Let us now consider the relation of our results for the spin-2 plus spin-4 string to the $c = \frac{4}{5}$ minimal models discussed above. The conformal weights of the primary fields of the lowest W_3 minimal model, which has $c = \frac{4}{5}$, are $h = \{0, \frac{1}{15}, \frac{2}{5}, \frac{2}{3}\}$ [18]. One can see from Table 1 that these are conjugate to the effective intercepts Δ of some of the physical states,

* In fact in this case, and presumably in general, it seems that Δ alone does not fully characterise the sectors. In the lowest W_3 minimal model, the fields with conformal weights 0 and $\frac{2}{5}$ have zero weight under the spin-3 primary current, whilst the fields with conformal weights $\frac{1}{15}$ and $\frac{2}{3}$ each occur twice, once with a positive, and once with an equal and opposite negative weight under the spin-3 current. This is reflected in our results for the physical states, where one can see evidence for two independent sequences of $\Delta = \frac{14}{15}$ states and two independent sequences of $\Delta = \frac{1}{3}$ states. Within each sequence, the φ momenta are such that they can be obtained from the lowest-level member by the action of the screening charge, but the φ momenta for the two sequences cannot be matched by any integer powers of the screening charge.

in the sense that $\Delta = 1 - h$. On the other hand, the $N = 5$ Virasoro minimal model also has $c = \frac{4}{5}$, and its primary fields have dimensions $h = \{0, \frac{1}{40}, \frac{1}{15}, \frac{1}{8}, \frac{2}{5}, \frac{21}{40}, \frac{2}{3}, \frac{7}{5}, \frac{13}{8}, 3\}$. We see from Table 1 that the full set of effective intercepts Δ is conjugate to a subset of the Virasoro minimal model weights, namely $1 - \Delta = h = \{0, \frac{1}{15}, \frac{2}{5}, \frac{2}{3}, \frac{7}{5}, 3\}$. In fact one can easily check from the fusion rules [19] for the $N = 5$ Virasoro minimal model that this particular subset of fields closes on itself. This provides an indication that the set of effective intercepts that we have found by studying levels up to $\ell = 9$ in the spin-2 plus spin-4 string is probably complete. In fact, part of our motivation in searching to levels as high as $\ell = 9$ was to find the “missing” intercept $\Delta = -2$, which does not make its appearance until this level. We expect that the three-point correlation functions of the physical operators of the spin-2 plus spin-4 string, calculated using the procedures introduced in [11,9], will be in agreement with the fusion rules of the closed subset of $c = \frac{4}{5}$ Virasoro minimal model fields listed above. Higher-point correlation functions should be consistent with these three-point functions. The spin-2 plus spin-4 string seems to admit the interpretation of a $c = 25\frac{1}{5}$ matter system with fields X^μ , coupled to a $c = \frac{4}{5}$ model realised by the $\{\varphi, \beta, \gamma\}$ system. Note that here, and for all the spin-2 plus spin- s strings that we are considering in this paper, the physical states can be divided into “prime states,” occurring at the lowest ghost number for a given level ℓ , and higher ghost number partners built by acting on the prime states with the ghost boosters $a_\varphi \equiv [Q_B, \varphi]$ and $a_{X^\mu} \equiv [Q_B, X^\mu]$. (All of the physical states listed in the Tables in this paper are prime states.) It is sometimes necessary to use the ghost boosters in order to obtain non-vanishing correlation functions (analogous to the use of picture changing in the superstring). The structure of the multiplets of states generated by the ghost boosters is discussed in detail in [11,9] for the W_3 string; the same considerations apply for the string theories in this paper.

For the spin-2 plus spin-5 string, with α given by (2.8) for $s = 5$, we have studied physical states up to and including level $\ell = 13$. We find $\ell = 0$ prime states with “standard” ghost structure, described by ghost number $G = 5$ physical operators of the form $U = c \partial^3 \gamma \partial^2 \gamma \partial \gamma \gamma e^{\mu\varphi} e^{ip \cdot X}$, with $\mu = -\frac{10}{11}\alpha, -\frac{12}{11}\alpha, -\frac{21}{22}\alpha, -\frac{23}{22}\alpha, -\alpha$. The corresponding values for the effective intercepts are $\Delta = 1, 1, \frac{15}{16}, \frac{15}{16}, \frac{11}{12}$. At level $\ell = 1$, we find prime states corresponding to $G = 4$ physical operators of the form $U = c \partial^2 \gamma \partial \gamma \gamma e^{\mu\varphi} e^{ip \cdot X}$, with $\mu = -\frac{8}{11}, -\frac{9}{11}, -\frac{15}{22}, -\frac{17}{22}$, and corresponding intercept values $\Delta = \frac{2}{3}, \frac{1}{4}, \frac{15}{16}, \frac{7}{16}$. The results up to level 13 are displayed in Table 2 below.

Again, we see that the φ momentum is always frozen to values that are integer multiples of a basic quantum, in this case $\mu = \frac{k}{22}\alpha$. From the mass-shell condition we therefore find that a necessary condition for the existence of a physical state with effective intercept Δ at level ℓ is that there exist an integer k such that

$$(k + 22)^2 = 48(\Delta + \ell) - 44. \quad (3.8)$$

Discrete states with zero momentum in the effective spacetime could therefore arise at levels $\ell = 1, 3, 5, 11, 15, 25, 31, \dots$. The $SL(2, C)$ vacuum is a discrete physical state at $\ell = 11$,

corresponding to the identity operator. The analogue of the ground-ring operator could be expected to arise at the next permitted level, namely $\ell = 15$ with φ momentum $\mu = \frac{2}{11}\alpha$. We have checked explicitly, and found that indeed the ghost number $G = -1$ operator

$$S = \beta e^{\frac{2}{11}\alpha\varphi} \quad (3.9)$$

is a screening current, with

$$\{Q_B, S\} = \partial D, \quad (3.10)$$

where D is the $\ell = 15$ discrete state. One should again be able to act with powers of the screening charge, built by integrating (3.9), on the lowest-level representative of each class of physical states, characterised in part by Δ , in order to generate all the higher-level ones. (As for the $s = 4$ case, there can be more than one class of states with a given Δ ; for example when $\Delta = \frac{15}{16}$.)

From (3.1) we see that in this case, where $s = 5$, the effective theory has central charge 25, conjugate to the central charge $c = 1$ of the lowest W_4 minimal model. Thus we should expect that a subset of the effective intercept values Δ should be conjugate to the weights

	G	Δ			μ (In units of $\alpha/22$)		
$\ell = 0$	5	$\frac{11}{12}$	$\frac{15}{16}$	1	-22	$\{-23, -21\}$	$\{-24, -20\}$
$\ell = 1$	4	$\frac{1}{4}$	$\frac{7}{16}$	$\frac{2}{3}$	$\frac{15}{16}$	-18	-17 -16 -15
$\ell = 2$	4	$-\frac{9}{16}$	$-\frac{1}{3}$	$\frac{1}{4}$		-17	-16 -14
$\ell = 3$	3	0	$\frac{7}{16}$	$\frac{11}{12}$		-12	-11 -10
$\ell = 4$	3		$-\frac{9}{16}$	$\frac{7}{16}$			-11 - 9
$\ell = 5$	3	-2	$-\frac{13}{12}$	$-\frac{9}{16}$	0	-12	-10 - 9 - 8
$\ell = 6$	2		$-\frac{25}{12}$	$\frac{15}{16}$			- 6 - 5
$\ell = 7$	2			$\frac{2}{3}$			- 4
$\ell = 8$	2		$-\frac{1}{3}$	$\frac{7}{16}$			- 4 - 3
$\ell = 9$	2	$-\frac{33}{16}$	$-\frac{9}{16}$	$\frac{1}{4}$		- 5	- 3 - 2
$\ell = 10$	1			1			0
$\ell = 11$	1			$\frac{15}{16}$			1
$\ell = 12$	1			$\frac{11}{12}$			2
$\ell = 13$	1		-2	$\frac{15}{16}$		0	3

Table 2. Continuous-momentum physical operators for the spin-2 plus spin-5 string

of this minimal model, namely $h = \{0, \frac{1}{16}, \frac{1}{12}, \frac{1}{3}, \frac{9}{16}, \frac{3}{4}, 1\}$ [20]. Indeed this is the case, as can be seen by looking at the results in Table 2. Unlike the spin-2 plus spin-4 string discussed above, where the central charge $c = \frac{4}{5}$ of the W_3 minimal model happened to coincide with that of a Virasoro minimal model, here the central charge $c = 1$ of the W_4 minimal model does not coincide with a standard Virasoro minimal model. On the other hand, it is known that there do exist “curiosities” at $c = 1$ in the Virasoro algebra [21], and it may well be that the set of weights $h = 1 - \Delta$ conjugate to the effective intercepts for the spin-2 plus

spin-5 string are those of such a curiosity. The weights that we have found, from Table 2, are: $1 - \Delta = h = \{0, \frac{1}{16}, \frac{1}{12}, \frac{1}{3}, \frac{9}{16}, \frac{3}{4}, 1, \frac{4}{3}, \frac{25}{16}, 3, \frac{49}{16}, \frac{37}{12}\}$. We do not know if this exhausts the list, or whether further values might arise if one looks at higher-level physical states.

An interesting feature of the spin-2 plus spin-5 string is that the central charge of the effective spacetime energy-momentum tensor T^{eff} takes an integer value, namely $c^{\text{eff}} = 25$. This means that in this case one avoids the necessity of including a background-charge vector a_μ in (2.7), by choosing $d = 25$. It may be that the resulting string theory can be viewed as a 25-dimensional Virasoro string coupled to a “curiosity at $c = 1$,” realised by the $\{\varphi, \beta, \gamma\}$ system.

For the spin-2 plus spin-6 string, with α given by (2.8) for $s = 6$, we have analysed the physical states up to level $\ell = 6$. Standard states correspond to $\ell = 0$ physical operators with ghost number $G = 6$, of the form $U = c \partial^4 \gamma \partial^3 \gamma \partial^2 \gamma \partial \gamma \gamma e^{\mu \varphi} e^{ip \cdot X}$. Our findings for the ghost numbers, effective intercepts and φ momenta for all levels up to and including 6 are presented in Table 3 below.

	G	Δ					μ (In the units of $\alpha/65$)				
$\ell = 0$	6			$\frac{32}{35}$	$\frac{33}{35}$	1	$\{-66, -64\}$	$\{-68, -62\}$	$\{-70, -60\}$		
$\ell = 1$	5	$\frac{1}{5}$	$\frac{12}{35}$	$\frac{18}{35}$	$\frac{5}{7}$	$\frac{33}{35}$	-56	-54	-52	-50	-48
$\ell = 2$	5		$-\frac{23}{35}$	$-\frac{17}{35}$	$-\frac{2}{7}$	$\frac{1}{5}$		-54	-52	-50	-46
$\ell = 3$	4		$-\frac{1}{5}$	$\frac{1}{7}$	$\frac{18}{35}$	$\frac{32}{35}$	-42	-40	-38	-36	
$\ell = 4$	4			$-\frac{6}{7}$	$-\frac{17}{35}$	$\frac{12}{35}$		-40	-38	-34	
$\ell = 5$	4	$-\frac{11}{5}$	$-\frac{13}{7}$	$-\frac{52}{35}$	$-\frac{38}{35}$	$-\frac{23}{35}$	-42	-40	-38	-36	-34
$\ell = 6$	3			$-\frac{1}{5}$	$\frac{12}{35}$	$\frac{32}{35}$			-28	-26	-24

Table 3. Continuous-momentum physical operators for the spin-2 plus spin-6 string

We see in this case that the φ momentum is frozen to values of the form $\mu = \frac{k}{65}\alpha$, where k is an integer. From the mass-shell condition, it therefore follows that a necessary condition for the existence of a physical state with effective intercept Δ at level ℓ is that there should exist an integer k such that

$$(k + 65)^2 = 280(\Delta + \ell) - 255. \quad (3.11)$$

Discrete states with zero momentum in the effective spacetime might therefore occur at levels $\ell = 1, 16, 21, 66, \dots$. The $SL(2, C)$ vacuum corresponds to the identity operator at $\ell = 16$, and the analogue of the ground-ring operator can be expected to arise at level $\ell = 21$, with φ momentum $\mu = \frac{2}{13}\alpha$. We have checked that indeed the $G = -1$ current

$$S = \beta e^{\frac{2}{13}\alpha\varphi} \quad (3.12)$$

is a screening current, satisfying

$$\{Q_B, S\} = \partial D, \quad (3.13)$$

with D being the level $\ell = 21$ discrete state. Again, the screening charge obtained by integrating (3.12) can be expected to generate higher-level physical states from the lowest-level representatives for each effective intercept Δ . (As in the lower- s examples, there can be more than one class of states for a given Δ . This occurs, for example, in the $\Delta = \frac{32}{35}$ and $\Delta = \frac{33}{35}$ sectors.)

The central charge for the effective energy-momentum tensor T^{eff} is, from (3.1), equal to $26 - \frac{8}{7}$ for this spin-2 plus spin-6 string. Thus it is conjugate to the central charge $c = \frac{8}{7}$ of the lowest W_5 minimal model. This model has primary fields with conformal weights $h = \{0, \frac{2}{35}, \frac{3}{35}, \frac{2}{7}, \frac{17}{35}, \frac{23}{35}, \frac{4}{5}, \frac{6}{7}, \frac{6}{5}\}$ [20]. From the list of effective intercepts Δ in Table 3, we see that the conjugate weights $h = 1 - \Delta$ include all of the W_5 minimal-model weights given above, together with $h = \{\frac{9}{7}, \frac{13}{7}, \frac{20}{7}, \frac{16}{5}, \frac{52}{35}, \frac{58}{35}, \frac{73}{35}, \frac{87}{35}\}$. It may well be that there are also further effective intercept values that will arise only at levels beyond $\ell = 6$. Presumably the full set of conjugate conformal weights h will be associated with some particular $c = \frac{8}{7}$ Virasoro model, as realised by the $\{\varphi, \beta, \gamma\}$ system. Little is known about such $c > 1$ models.

4. Unitarity, and the other BRST operators

In section 3 we concentrated on the spin-2 plus spin- s BRST operators for which the background-charge parameter α is given by (2.8). These cases presumably correspond to truncations of the W_s algebra in which all but the spin-2 and spin- s currents are omitted. It seems that the corresponding string theories are likely to be unitary, in the sense that the effective intercept values Δ will not give rise to non-unitary states of the effective Virasoro string theory described by T^{eff} with central charge given by (3.1). For example, one can derive the following limits on the intercept values for a Virasoro string with central charge c , by requiring that level-1 and level-2 excited states have no negative-norm degrees of freedom:

$$\text{level 1 : } \quad \Delta \leq 1, \quad (4.1a)$$

$$\text{level 2 : } \quad \Delta \leq \frac{37-c-\sqrt{(c-1)(c-25)}}{16}, \quad \text{or} \quad \Delta \geq \frac{37-c+\sqrt{(c-1)(c-25)}}{16}. \quad (4.1b)$$

For the case of the spin-2 plus spin-4 string, this means that unitarity requires that $\Delta \leq \frac{3}{5}$ or $\frac{7}{8} \leq \Delta \leq 1$. From the results in Table 1, we see that these conditions are satisfied by all the sectors of the theory. Whilst this does not constitute a complete proof of unitarity, it certainly provides a strong indication that it holds for the spin-2 plus spin-4 theory. For the spin-2 plus spin-5 string, where $c^{\text{eff}} = 25$, the conditions (4.1a–b) reduce to $\Delta \leq 1$, which is satisfied by all the sectors of physical states listed in Table 2. For the spin-2 plus spin-6 string, and indeed for all cases with $s \geq 6$, the value of c^{eff} is less than 25. Under these circumstances, there is no unitarity restriction coming from level-2 excited states. The fact

that all the sectors of physical states given in Table 3 for the spin-2 plus spin-6 string have $\Delta \leq 1$ indicates that this theory is probably unitary too. Presumably this persists for all the higher spin-2 plus spin- s strings, in the case that α is given by (2.8).

The story may be different for the solutions for spin-2 plus spin- s BRST operators corresponding to other values of α that are not given by (2.8). For example, we found a second spin-2 plus spin-4 BRST operator, with Q_1 given by (2.3) and (2.11), with $\alpha^2 = \frac{361}{30}$. Solving for physical states, we find that there are standard ghost-structure states at $\ell = 0$ with $\mu = -\frac{17}{19}, -\frac{21}{19}, -\frac{18}{19}, -\frac{20}{19}$ and effective intercepts $\Delta = \frac{21}{20}, \frac{21}{20}, 1, 1$ respectively. We have checked all levels up to and including $\ell = 6$, and we find the following set of effective intercepts:

$$\Delta = \{\frac{21}{20}, 1, \frac{4}{5}, \frac{1}{4}\}. \quad (4.2)$$

The fact that one of the intercept values exceeds 1 is suggestive of non-unitarity of the physical states in this sector of the effective string theory. In fact, $c^{\text{eff}} = 26\frac{3}{5} > 26$, and so it would perhaps not be surprising if there were some difficulty with unitarity in this case. However, it is worth noting that the value of the intercept that exceeds 1, namely $\frac{21}{20}$, coincides with the limit of the second inequality in (4.1b); in other words, there would be a null state in the spectrum of physical states with level 2 excitations in the effective spacetime. It may be that a 2-scalar model, *i.e.* φ together with just one extra scalar X , would make sense even in a case where intercepts greater than 1 occur.

Similar remarks apply to the other BRST operators that we have found for the spin-2 plus spin-6 system. The case $\alpha^2 = \frac{361}{12}$ is potentially interesting because it corresponds to $c^{\text{eff}} = 26$. We have looked at the $\ell = 0$ physical states, and found the effective intercepts $\Delta = \{\frac{26}{25}, 1, \frac{24}{25}\}$. At $\ell = 1$, we find $\Delta = \{1, \frac{21}{25}, \frac{14}{25}, \frac{11}{25}, \frac{4}{25}\}$. For $\alpha^2 = \frac{1681}{56}$, we find $\ell = 0$ intercepts $\Delta = \{\frac{15}{14}, \frac{115}{112}, 1\}$, and for $\alpha^2 = \frac{5041}{168}$, we find $\ell = 0$ intercepts $\Delta = \{\frac{15}{14}, \frac{117}{112}, 1\}$.

5. Conclusions

We have given the general form for BRST operators for a class of extensions of the Virasoro algebra, where there is a primary current of spin s in addition to the energy-momentum tensor. We have found the explicit forms for the BRST operators in the cases $s = 4, 5$ and 6 . The case $s = 3$ corresponds to the well-known W_3 algebra, which has been well studied over the last few years.

One may use these BRST operators to build anomaly-free extensions of two-dimensional gravity, and hence to build extensions of ordinary string theory. We have discussed the detailed structure of the physical states for the cases $s = 4, 5$ and 6 . For the class of spin-2 plus spin- s BRST operators for which the background charge α for φ is given by (2.8), the effective spacetime energy-momentum tensor T^{eff} has the a central charge c^{eff} that is conjugate to that of the lowest W_{s-1} minimal model, in the sense that $26 = c^{\text{eff}} + c^{\text{min}}$. One expects therefore that the effective intercept values Δ for the physical states should be

conjugate to a set of weights h that at least include those of the associated W_{s-1} minimal model, in the sense that $1 = \Delta + h$. This is indeed what we have found, in the explicit examples of $s = 4, 5$ and 6 . However, because the physical states are subject to constraints only from a spin-2 and a spin- s current, there are more sectors than just those that are conjugate to the primary fields of the W_{s-1} minimal model. When $s = 4$, for which $c^{\text{eff}} = 26 - \frac{4}{5}$, we find intercept values conjugate to the weights of a subset of the fields of the 3-state Potts model that close under the fusion rules. For $s = 5$, where $c^{\text{eff}} = 26 - 1$, we find intercepts conjugate to the weights of the primary fields of a $c = 1$ Virasoro model. For $s = 6$, and higher, the intercepts are conjugate to models with $c > 1$. The spin-2 plus spin- s BRST operators with the background charge α given by (2.8) all appear to give rise to string theories that are unitary in the effective spacetime.

For the examples of spin-2 plus spin-4, and spin-2 plus spin-6 BRST operators, we have also found other solutions for which the central charge is not given by (2.8). Such solutions presumably exist for higher values of s too. These BRST operators appear to give rise to string theories that are not unitary from the effective spacetime point of view; in particular, one finds sectors of the theory with effective intercepts that exceed 1.

In a sense, all of the W -string theories that have been constructed suffer from the drawback that they can be reinterpreted as special cases of ordinary Virasoro strings in which one builds a $c = 26$ energy-momentum tensor as the direct sum of a spacetime energy-momentum tensor T^{eff} and an energy-momentum tensor T^{min} for a minimal model or some other model containing a finite number of primary fields. On the other hand, one can take the view that this close connection between minimal models and W strings reveals an interesting underlying W symmetry as an organising principle for the physical states.

For the W_3 string it is known that the structure of physical states in the two-scalar realisation is richer than in the multi-scalar realisations analogous to those that we have been considering in this paper. Specifically, one finds that the physical states of the multi-scalar W_3 string can all be understood as generalisations of some of the two-scalar physical states, but that the two-scalar theory also has further physical states that do not generalise beyond two dimensions [10,11,9]. A similar phenomenon occurs for the two-scalar spin-2 plus spin- s strings that we have been considering in this paper. For example, we find that there is an $\ell = 3$ physical operator in the two-scalar spin-2 plus spin-4 string, with $\alpha^2 = \frac{243}{20}$, of the form

$$U = \left(c\gamma + \frac{4}{3}\partial^2\gamma\gamma - \frac{16}{9}\alpha\partial\varphi\partial\gamma\gamma - \frac{8}{11}a\partial X\partial\gamma\gamma - \frac{8}{5}bc\partial\gamma\gamma \right) e^{-\frac{2}{3}\alpha\varphi + \frac{6}{11}aX}, \quad (5.1)$$

where a is the background charge for the single extra field X , and $a^2 = \frac{121}{60}$. This operator does not generalise to the multi-scalar case, when X is replaced by X^μ . It may be that just as for the W_3 string, the more subtle aspects of the underlying W algebra are better captured by the two-scalar than the multi-scalar realisations.

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